INTERPOLATION OF SUBSPACES AND APPLICATIONS TO EXPONENTIAL BASES IN SOBOLEV SPACES

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ABSTRACT. We give precise conditions under which the real interpolation space $[Y_0, X_1]_{\theta,p}$ coincides with a closed subspace of $[X_0, X_1]_{\theta,p}$ when Y_0 is a closed subspace of codimension one. We then apply this result to nonharmonic Fourier series in Sobolev spaces $H^s(-\pi, \pi)$ when $0 < s < 1$. The main result: let $\mathcal{E}$ be a family of exponentials $\exp(i\lambda_n t)$ and $\mathcal{E}$ forms an unconditional basis in $L^2(-\pi, \pi)$. Then there exist two number s_0, s_1 such that $\mathcal{E}$ forms an unconditional basis in H^s for $s < s_0$, $\mathcal{E}$ forms an unconditional basis in its span with codimension 1 in H^s for $s_1 < s$. For $s_0 \le s \le s_1$ the exponential family is not an unconditional basis in its span.

1. INTRODUCTION

In this paper we will apply a result on interpolation of subspaces to the study of exponential Riesz bases in Sobolev spaces.

In section 2 we consider the comparison of the interpolation spaces $X_\theta := [X_0, X_1]_{\theta,p}$ and $Y_\theta := [Y_0, X_1]_{\theta,p}$ for $1 \le p < \infty$, where Y_0 is a subspace of X_0 with codimension one, say $Y_0 = \ker \psi$ where $\psi \in X_0^*$. This problem, as far as we know, was first formulated in [16], v.1, Ch.1.18 in 1968. As we show in Theorem 2.1 there are two indices $0 \le \sigma_0 \le \sigma_1 \le 1$ which may be explicitly evaluated in terms of the K-functional of ψ so that:

1. If $0 < \theta < \sigma_0$ then Y_θ is a closed subspace of codimension one in X_θ.
2. If $\sigma_1 < \theta < 1$ then $Y_\theta = X_\theta$ with equivalence of norm and
3. If $\sigma_0 \le \theta \le \sigma_1$ then the norm on Y_θ is not equivalent to the norm on X_θ.

Let us discuss the history of this theorem. The special case of a Hilbert space of Sobolev type connected with elliptical boundary data was considered in [16], and in this case the critical indices σ_0 and σ_1 coincide. In the well known case [16] $X_1 = L^2(0, \infty)$, $X_0 = W_2^1(0, \infty)$ and Y_0 is the subspace of W_2^1 of functions vanishing at the origin, this critical value is $\sigma_0 = \sigma_1 = 1/2$. Later R. Wallsten [26] gave an example where the critical indices satisfy $\sigma_0 < \sigma_1$. The general problem was considered by J. Löfström [17], where some special cases of Theorem 2.1 are obtained. Later, Löfström in an unpublished (but

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web-posted) preprint from 1997, obtained most of the conclusion of Theorem 2.1: specifically he obtained the same result except he did not treat the critical values $\theta = \sigma_0, \sigma_1$. The authors were not aware of Löfström's earlier work during the initial preparation of this article and our approach is rather different. A more general but closely related problem on interpolating subspaces of codimension one has been recently considered in [12] and [10]. For general results on subcouples we refer to [9].

Let us recall next that a sequence $(e_n)_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz basic sequence* if there is a constant C so that for any finitely non-zero sequence $(a_n)_{n\in\mathbb{Z}}$ we have

$$\frac{1}{C}\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right)^{\frac{1}{2}} \leq \|\sum_{n\in\mathbb{Z}} a_n e_n\| \leq C\left(\sum_{n\in\mathbb{Z}}|a_n|^2\right)^{\frac{1}{2}}.$$

A *Riesz basis* for $\mathcal{H}$ is a Riesz basic sequence whose closed linear span $[e_n]_{n\in\mathbb{Z}} = \mathcal{H}$. A sequence (e_n) is an *unconditional basis,* respectively *unconditional basic sequence* if $(e_n/\|e_n\|)_{n\in\mathbb{Z}}$ is a Riesz basis, respectively, a Riesz basic sequence.

In the second part of the paper we apply our interpolation result to study the basis properties of exponential families $\{e^{i\lambda_n t}\}$ in Sobolev spaces. These families appear in such fields of mathematics as the theory of dissipative operators (the Sz.–Nagy–Foias model) , the Regge problem for resonance scattering, the theory of initial boundary value problems, control theory for distributed parameter systems, and signal processing, see, e.g., [22], [8], [13], [2], [25]. One of the most important problems arising in all of these applications is the question of the Riesz basis property od these families. In the space $L^2(-\pi,\pi)$ this problem has been studied for the first time in the classical work of Paley and Wiener [23]. The problem has now a complete solution [11], [20] on the basis of an approach suggested by B. S. Pavlov.

The principal result for Riesz bases can be formulated as follows [11].

Proposition 1.1. *The sequence $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ if and only if* $\sup|\Im\lambda_n| < \infty$,

$$\inf_{k\neq j}|\lambda_k - \lambda_j| > 0. \tag{1.1}$$

and there is an entire function F of exponential type π (the generating function) with simple zeros at $(\lambda_n)_{n\in\mathbb{Z}}$ and such that for some y $|F(x+iy)|^2$ satisfies the Muckenhoupt condition (A_2) (we shall write this as $|F|^2 \in (A_2)$):

$$\sup_{I\in\mathcal{J}}\left\{\frac{1}{|I|}\int_I |F(x+iy)|^2\,dx\;\frac{1}{|I|}\int_I |F(x+iy)|^{-2}\,dx\right\} < \infty,$$

where $\mathcal{J}$ is the set of all intervals of the real axis.

In [20] a corresponding characterization is given for exponential families which form an unconditional basis of $L^2(-\pi,\pi)$ when $\Im\lambda_n$ can be unbounded both from above and below.

Let us describe known results concerning exponential bases in Sobolev spaces. The first result in this direction has been obtained by D. L. Russell in [24] Russell studied the unconditional basis property for exponential families in the Sobolev spaces $H^m(-\pi,\pi)$ with $m\in\mathbb{Z}$.

Proposition 1.2. [24] *Suppose $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$. Suppose $m\in\mathbb{N}$ and suppose $\mu_1,\cdots,\mu_m\in\mathbb{C}\setminus\{\lambda_n:\ n\in\mathbb{Z}\}$ are distinct. Then $(e^{i\lambda_n t})_{n\in\mathbb{Z}}\cup(e^{i\mu_k t})_{k=1}^m$ is an unconditional basis of $H^m(-\pi,\pi)$. In particular $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ is an unconditional basic sequence whose closed linear span has codimension m in $H^m(-\pi,\pi)$.*

In [21] the unconditional basis property for an exponential family was studied in $H^s(-\pi,\pi)$ for noninteger s for the case λ_n being the eigenvalues of a Sturm–Liouville operator with a smooth potential.

Note that the generalization of the Levin–Golovin theorem for Sobolev spaces has been obtained [3] using 'classical methods' of the entire function theory. Suppose $\{\lambda_n\}_{n\in\mathbb{Z}}$ are the zeros of an entire function F of exponential type π, (λ_n) is separated (1.1), and on some line $\{x+iy\}_{x\in\mathbb{R}}$ we have

$$C^{-1}(1+|x|)^s\le|F(x+iy)|\le C(1+|x|)^s.$$

Then the family $\{e^{i\lambda_k t}/(1+|\lambda_k|)^s\}$ forms a Riesz basis in $H^s(-\pi,\pi)$. Notice that this result was applied to several controllability problems for the wave type equation [4].

Recently Yu. Lyubarskii and K. Seip [19] have established a necessary and sufficient criterion for sampling/interpolation problem for weighted Paley-Wiener spaces, which gives a criterion for a sequence to be an unconditional basis in H^s. For the case, $\sup|\Im\lambda_n|<\infty$, the main result is the following:

Theorem 1.3. *$(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ forms an unconditional basis in $H^s(-\pi,\pi)$ if and only if (λ_n) is separated (i.e. (1.1) holds) and for the generating function F we have $|F(x+iy)|^2/(1+|x|^{2s})\in(A_2)$ for some y.*

The main idea of the present paper is that if $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(-\pi,\pi)$ then it also forms an unconditional basis of a subspace Y_0 of $H^1(-\pi,\pi)$ of codimension one. Then, by interpolation, one obtains that $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ is an unconditional basis of the intermediate spaces $[Y_0,L^2]_{\theta,2}$ for $0<\theta<1$. This approach was suggested in [7] by the first author. The main result of [7] is incorrect in the general case because a mistake connected with interpolation of subspaces. Here we correct this mistake.

Let us describe the results concerning unconditional bases in Sobolev spaces. One of our main results for Riesz bases is as follows:

Theorem 1.4. *Suppose $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ forms a Riesz basis of $L^2(-\pi,\pi)$. Suppose $(\lambda_n - n)_{n\in\mathbb{Z}}$ is bounded and let $\delta_n = \Re\lambda_n - n$. Then there exist critical indices $0 < s_0 \le s_1 < 1$ given by:*

$$s_1 = \frac{1}{2} - \lim_{\tau\to\infty} \inf_{t\ge 1} \frac{1}{\log\tau} \sum_{t<|n|\le\tau t} \frac{\delta_n}{n}$$

and

$$s_0 = \frac{1}{2} - \lim_{\tau\to\infty} \sup_{t\ge 1} \frac{1}{\log\tau} \sum_{t<|n|\le\tau t} \frac{\delta_n}{n}$$

such that:
(1) $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of the Sobolev space H^s if and only if $0 \le s < s_0$.
(2) $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of a closed subspace of H^s of codimension one if and only if $s_1 < s \le 1$.
(3) If $s_0 \le s \le s_1$ then (e_{λ_n}) is not an unconditional basic sequence.

This result is deduced from results in Sections 3 and 4. In Section 4 we in fact consider the more general situation for unconditional bases and give rather more technical results. The above Theorem 1.4 however is the simplest case and follows by combining Theorem 4.2, Theorem 4.9 and Theorem 4.10. Our approach is based on estimates of the K-functional for the continuous linear functional on $H^1(-\pi,\pi)$ which annihilates each $e^{i\lambda_n x}$ whose existence is guaranteed by the result of Russell (Proposition 1.2). The estimates are in terms of the generating function F.

Once one has Theorem 1.4 then it is easy to construct real sequences (λ_n) to show that s_0, s_1 can take any values in $(0,1)$ such that $s_0 \le s_1$. In the case of regular power behavior of F i.e. for some $y \ge 0$, $|F(x+iy)| \sim (1+|x|)^s$ one has $s_1 = s_0 = s + \frac{1}{2}$.

The results for the whole scale $H^s(-\pi,\pi)$ can then be obtained by 'shift' using the fact that the differentiation operator with appropriate conditions is an isomorphism between a one-codimensional subspace of H^m and H^{m-1}; we will not pursue this extension.

2. Interpolation of subspaces

Let (X_0, X_1) be a Banach couple with $X_0 \cap X_1$ dense in X_0, X_1. If $0 < \theta < 1$ and $1 \le p < \infty$ the real interpolation space $X_\theta = [X_0, X_1]_{\theta,p}$ is defined, see, e.g., [5], to be the set of all $x \in X_0 + X_1$ such that

$$\|x\|_{X_\theta} = \left(\int_0^\infty t^{\theta p-1} K(t,x)^p dt\right)^{\frac{1}{p}} < \infty,$$

where $K(t,x)$ is the K–functional. An equivalent definition [5] p. 314 (yielding an equivalent norm) can be given by using the J-method:

$$\|x\|_{X_\theta} = \inf\left\{ \left(\sum_{k\in\mathbb{Z}} \max\{\|x_k\|_0, 2^k\|x_k\|_1\}^p \right)^{\frac{1}{p}} \;:\; x = \sum_{k\in\mathbb{Z}} 2^{\theta k} x_k \right\},$$

where the series converges in $X_0 + X_1$.

Now suppose $0 \neq \psi \in X_0^*$ and let Y_0 be its kernel. We suppose also (only this case is interesting) that $Y_0 \cap X_1$ is dense in X_1, i.e., ψ is not bounded in X_1.

Let Y_θ be the corresponding spaces obtained by interpolating Y_0 and X_1. Clearly $Y_\theta \subset X_\theta$ and the inclusion has norm one. It is easy to show that the closure of Y_θ in X_θ is either a subspace of codimension one when ψ is continuous on X_θ or the whole of X_θ when ψ is not continuous.

Let us now introduce two important indices.

$$\sigma_1 = \lim_{\tau\to\infty} \sup_{0<\tau t\leq 1} \frac{1}{\log\tau} \log\frac{K(\tau t,\psi)}{K(t,\psi)}$$

and

$$\sigma_0 = \lim_{\tau\to\infty} \inf_{0<\tau t\leq 1} \frac{1}{\log\tau} \log\frac{K(\tau t,\psi)}{K(t,\psi)},$$

where $K(t,\psi) = K(t,\psi; X_0^*, X_1^*)$. ¿From the multiplicative properties of the function $K(\tau t,\psi)/K(t,\psi)$ it is clear that these limits exist and $0 \leq \sigma_0 \leq \sigma_1 \leq 1$. Since $K(t,\psi)$ is bounded as $t \to \infty$ we can also write:

$$\sigma_1 = \lim_{\tau\to\infty} \sup_{0<t<\infty} \frac{1}{\log\tau} \log\frac{K(\tau t,\psi)}{K(t,\psi)}.$$

Let us observe that:

$$\sup\{|\psi(x)| :\; \max\{\|x\|_0, t\|x\|_1\} \leq 1\} = K(t^{-1},\psi)$$

We define a sequence $(w_n)_{n\in\mathbb{Z}}$ by

$$w_n = K(2^{-n},\psi)^{-1}.$$

Notice that $\inf_{n\in\mathbb{Z}} w_n \geq \|\psi\|_{X_0^*}^{-1} > 0$ and that in general $w_n \leq w_{n+1} \leq 2w_n$. Now it is easy to see that

$$\begin{aligned} \sigma_1 &= \lim_{k\to\infty} \sup_n \tfrac{1}{k} \log_2 \tfrac{w_{n+k}}{w_n} \\ \sigma_0 &= \lim_{k\to\infty} \inf_{n\geq 0} \tfrac{1}{k} \log_2 \tfrac{w_{n+k}}{w_n}. \end{aligned}$$

As mentioned in the introduction the following result is a slight improvement of a result of Löfström [18], who obtains the same result by quite different arguments except for the critical indices $\theta = \sigma_0, \sigma_1$.

Theorem 2.1. *1. $Y_\theta = X_\theta$ (with equivalence of norm) if and only if $\theta > \sigma_1$.*
2. Y_θ is a closed subspace of codimension one in X_θ if and only if $\theta < \sigma_0$.
3.If $\sigma_0 \le \theta \le \sigma_1$ then Y_θ is not closed in X_θ.

We shall consider the weighted ℓ_p space $\ell_p(w)$ of all sequences $(\alpha_n)_{n\in\mathbb{Z}}$ such that

$$\|\alpha\| = \left(\sum_{k\in\mathbb{Z}} w_n^p |\alpha_n|^p\right)^{\frac{1}{p}}.$$

We shall use ζ_n for the standard basis vectors. On $\ell_p(w)$ we consider the shift operator $S((\alpha_n)) = (\alpha_{n-1})$. From the above remarks it is clear that S, S^{-1} are both bounded and $\|S\| \le 2, \|S^{-1}\| = 1$. Furthermore the spectral radius formula shows that 2^{σ_1} is the spectral radius $r(S)$ of S. Now let P_+ be the projection $P_+(\alpha) = (\delta_n \alpha_n)$ where $\delta_n = 1$ if $n \ge 0$ and 0 otherwise. It is easy to calculate

$$\|P_+ S^{-n}\| = \sup_{k\ge 0} \frac{w_k}{w_{n+k}}$$

and so this implies that $r(P_+S^{-1}) = 2^{-\sigma_0}$.

We will need the following key Lemma:

Lemma 2.2. *Let $0 < \theta < 1$ and let $T_\theta = S - 2^\theta I$. Then*

1. *T_θ is an isomorphism onto $\ell_p(w)$ if and only if $\sigma_1 < \theta$.*
2. *T is an isomorphism onto a proper closed subspace if and only if $\theta < \sigma_0$. In this case the range of T is the subspace of codimension one of all α such that $\sum_{n\in\mathbb{Z}} 2^{n\theta}\alpha_n = 0$.*

Proof. First observe that if $\theta > \sigma_1$ then T_θ must be an isomorphism onto $\ell_p(w)$ since 2^θ exceeds the spectral radius of S. Furthermore since the spectrum of S is invariant under rotations it is clear that T_{σ_1} cannot be an isomorphism onto $\ell_p(w)$. Also note that T_θ is always injective and that if f_θ is a linear functional annihilating its range then $f_\theta(\zeta_n) = c2^{n\theta}$ for some constant c, i.e., $f_\theta(\alpha) = \sum_{n\in\mathbb{Z}} 2^{n\theta}\alpha_n = 0$. This implies that the closure of the range is either the whole space or the subspace of codimension one when $\sum_{n\in\mathbb{Z}} 2^{n\theta q} w_n^{-q} < \infty$. Here $\frac{1}{p} + \frac{1}{q} = 1$ and the formula must be modified if $p = 1$.

We next show that if $\theta < \sigma_0$ then T_θ is an isomorphism onto a closed subspace of codimension one.

Next let $E = [\{\zeta_n : \ n \le -1\}]$ and $F = [\{\zeta_n : \ n \ge 1\}]$. We remark that $T_\theta(E)$ is easily seen to be closed because T_θ is an isomorphism on the *unweighted* ℓ_p and w_n is bounded for $n \le -1$. If we show $T_\theta(F)$ is closed then we are done, since it is clear this will imply that $T_\theta(E+F)$ is closed and this is a subspace of co-dimension one in the range. However $2^{-\theta} > r(P_+S^{-1})$ so that $2^{-\theta} - P_+S^{-1}$ is an isomorphism. Restricting to F this implies $(2^{-\theta} - S^{-1})F$ and hence $T_\theta(F)$ is closed.

The proof is completed by showing that if $\theta \leq \sigma_1$ then if T_θ if has closed range it must satisfy $\theta < \sigma_0$. Note first that it is enough to establish this for $\theta < \sigma_1$ since the set of operators with Fredholm index one is open. Suppose $\sigma_0 < \theta < \sigma_1$ and T_θ is closed. Then T_θ has a lower estimate $\|T_\theta\alpha\| \geq c\|\alpha\|$ for all α where $c > 0$. Assume $w_{n+k} > 2^{n\theta}w_k$ for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then consider $\alpha = (I + 2^{-\theta}S + \cdots + 2^{-n\theta}S^n)^2\zeta_k$. Note that $\|\alpha\| \geq n2^{-n\theta}w_{n+k}$. However

$$\begin{aligned}\|T_\theta^2\alpha\| =& 2^{2\theta}w_k + 2\cdot 2^{(-n+1)\theta}w_{n+k+1} + 2^{-2n\theta}w_{2n+k+2}\\ &\leq 8\max\{w_k, 2^{-n\theta}w_{n+k}, 2^{-2n\theta}w_{2n+k}\}.\end{aligned}$$

Let $v_n = 2^{-n\theta}w_n$. Then we have if $nc^2 > 8$,

$$(nc^2 - 8)v_{k+n} \leq 8\max\{v_k, v_{k+2n}\}.$$

In particular if $nc^2 > 16$,

$$v_{k+n} < \max\{v_k, v_{k+2n}\}.$$

Now since $\theta < \sigma_1$, we can find $k \in \mathbb{Z}, n > 16c^{-2}$ so that $w_{n+k} < 2^{n\theta}w_k$ or $v_{n+k} < v_k$. Iterating gives us that $(v_{k+rn})_{r=0}^\infty$ is monotone increasing. Now for any large N and any $j \geq 0$ we have

$$\frac{w_{j+N}}{w_j} \geq \frac{w_{k+r_2n}}{w_{k+r_1n}} \geq 2^{n(r_2-r_1)\theta}$$

where r_1, r_2 such that $k + (r_1 - 1)n \leq j \leq k + r_1n$ and $k + r_2n \leq j + N \leq k + (r_2 + 1)n$. This gives us

$$\frac{w_{j+N}}{w_j} \geq 2^{(N-2n)\theta}.$$

Hence

$$\inf_{j\geq 0}\frac{1}{N}\log_2\frac{w_{j+N}}{w_j} \geq (1 - \frac{2n}{N})\theta.$$

Letting $N \to \infty$ gives $\sigma_0 \geq \theta$. To show that in fact $\theta < \sigma_0$ needs only the observation again that the set of θ where T_θ has Fredholm index one is open.

□

We now use Lemma 2.2 to establish our main result Theorem 2.1 on interpolating subspaces:

Proof. Let us suppose next that either (a) $\theta < \sigma_0$ or (b) $\theta > \sigma_1$. This implies there exists a constant D so that $\|\alpha\| \leq D\|T_\theta\alpha\|$ for all $\alpha \in \ell_p(w)$; in case (a) T_θ maps onto the subspace of $\ell_p(w)$ defined by $f_\theta(\alpha) = \sum_{n\in\mathbb{Z}} 2^{n\theta}\alpha_n = 0$, while in case (b) T_θ is an isomorphism onto the whole space (see Lemma 2.2). We observe that in case (a) the linear functional ψ extends to a continuous linear functional on X_θ as

$$\sum_{n\in\mathbb{Z}} 2^{n\theta}K(2^n, \psi) < \infty.$$

Now suppose $x \in X_\theta$ with $\|x\|_{X_\theta} = 1$ with the additional assumption in case (a) that $\psi(x) = 0$. Then we may find $(x_n)_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} 2^{\theta n} x_n = x$ and

$$\left(\sum_{k\in\mathbb{Z}} \max\{\|x_k\|_0, 2^k\|x_k\|_1\}^p\right)^{\frac{1}{p}} \leq 2.$$

Then

$$\left(\sum_{n\in\mathbb{Z}} |\psi(x_n)|^p w_n^p\right)^{\frac{1}{p}} \leq 2,$$

since

$$|\psi(x)| \leq w_n^{-1} \max\{\|x\|_0, 2^n\|x\|_1\}.$$

In case (a) we additionally have

$$\sum_{n\in\mathbb{Z}} 2^{n\theta}\psi(x_n) = 0.$$

Thus we can find $\alpha \in \ell_p(w)$ with $T_\theta(\alpha) = (\psi(x_n))$ and $\|\alpha\| \leq 2D$. Then we can find $u_n \in X_0 \cap X_1$ such that $\max\{\|u_n\|_0, 2^n\|u_n\|_1\} \leq 2|\alpha_n|w_n$ and $\psi(u_n) = \alpha_n$. Let $v_n = u_{n-1} - 2^\theta u_n$. Then

$$\left(\sum_{k\in\mathbb{Z}} \max\{\|v_k\|, 2^k\|v_k\|_1\}^p\right)^{\frac{1}{p}} \leq 16D\|x\|_{X_\theta}.$$

Now $\psi(v_n) = \alpha_{n-1} - 2^\theta\alpha_n = \psi(x_n)$ and $\sum_{n\in\mathbb{Z}} 2^{n\theta} v_n = 0$. Hence

$$x = \sum_{n\in\mathbb{Z}} 2^{\theta n}(x_n - v_n)$$

and so $x \in Y_\theta$ with $\|x\|_{Y_\theta} \leq (16D+2)\|x\|_{X_\theta}$. From this it follows that in case (a) we have $Y_\theta = \{x : \ \psi(x) = 0, \ x \in X_\theta\}$ and in case (b) $Y_\theta = X_\theta$.

Next we consider the converse directions. Assume either (aa) ψ is continuous on X_θ and $Y_\theta = \{x : \ \psi(x) = 0, \ x \in X_\theta\}$ or (bb) $Y_\theta = X_\theta$. In either case there is a constant D so that if $x \in Y_\theta$ then $\|x\|_{Y_\theta} \leq D\|x\|_{X_\theta}$. Observe that in case (a) the linear functional f_θ is continuous on $\ell_p(w)$ and so the range of T_θ is contained in its kernel; in case (bb) its range is dense.

Assume $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \in \ell_p(w)$ with $\|\alpha\| = 1$; in case (aa) we also assume $f_\theta(\alpha) = 0$. We first find $x_n \in X_0 \cap X_1$ with $\psi(x_n) = \alpha_n$ and so that $\max\{\|x_n\|_0, 2^n\|x_n\|\} \leq 2|\alpha_n|w_n$ for $n \in \mathbb{Z}$. Let $x = \sum_{n\in\mathbb{Z}} 2^{n\theta}x_n$ so that $x \in X_\theta$ with $\|x\|_{X_\theta} \leq 2$. In case (aa) we have additionally that $\psi(x) = f_\theta(\alpha) = 0$. Now we can find $y_n \in Y_0 \cap X_1$ so that $\sum_{n\in\mathbb{Z}} 2^{n\theta} y_n = x$ and

$$\left(\sum_{k\in\mathbb{Z}} \max\{\|y_k\|, 2^k\|y_k\|_1\}^p\right)^{\frac{1}{p}} \leq 4D.$$

Now let $u_n = x_n - y_n$ and $v_n = \sum_{k=n+1}^{\infty} 2^{(k-n-1)\theta} u_k$. Then

$$\left(\sum_{k\in\mathbb{Z}} \max\{\|u_k\|, 2^k\|u_k\|_1\}^p\right\}^{\frac{1}{p}} \leq 4D + 2.$$

We argue that

$$\left(\sum_{k\in\mathbb{Z}} \max\{\|v_k\|_0, 2^k\|v_k\|_1\}^p\right)^{\frac{1}{p}} \leq C_\theta(4D+2), \tag{2.1}$$

where

$$C_\theta = (\sum_{k<0} 2^{k\theta} + \sum_{k\geq 0} 2^{k(\theta-1)}).$$

To show (2.1) we note that

$$2^n\|v_n\|_1 \leq \sum_{k=n+1}^{\infty} 2^{(k-n-1)(\theta-1)} 2^k \|u_k\|_1$$

and (since $\sum 2^{n\theta} u_n = 0$)

$$\|v_n\|_0 \leq \sum_{k=-\infty}^{n} 2^{(k-n-1)\theta} \|u_k\|_0.$$

Let $\beta_n = \psi(v_n)$. Then $\beta \in \ell_p(w)$ and $\|\beta\| \leq C_\theta(4D+2)$. But now $(T_\theta(\beta))_n = \psi(u_n) = \psi(x_n) = \alpha_n$ so that T_θ is an isomorphism onto the kernel of f_θ in case (aa) or onto $\ell_p(w)$ in case (bb). These two cases combined with the observation that Y_θ can only be a proper closed subspace of X_θ if ψ is continuous on X_θ complete the proof of the Theorem. $\square$

3. Sobolev spaces

In this section we investigate a special case of the results of the previous section for Sobolev spaces. These results are preparatory for Section 4 where we apply them to exponential bases. Let $L^2 = L^2(-\pi, \pi)$ and let us denote the standard inner-product on $L^2(-\pi, \pi)$ by

$$(f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

We denote by $\|f\|$ the standard norm on L^2.

For $s > 0$ we define the Sobolev space $H^s(\mathbb{R})$ to be the space of all $f \in L_2(\mathbb{R})$ so that

$$\|f\|_{H^s}^2 := \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2(1 + |\xi|^{2s})d\xi < \infty$$

($\hat{f}$ is the Fourier transform). We then define the Sobolev space $H^s = H^s(-\pi, \pi)$ to be the space of restrictions of $H^s(\mathbb{R})-$functions to the interval $(-\pi, \pi)$ (with

the obvious induced quotient norm). When $s = 1$ the space H^1 reduces to the space of $f \in L^2(-\pi, \pi)$ so that $f' \in L^2$ under the (equivalent) norm:

$$\|f\|_1^2 = \int_{-\pi}^{\pi} |f(t)|^2 + |f'(t)|^2 dt < \infty.$$

Then if $0 < s < 1$ we have $H^s = [H^1, L^2]_{1-s} = [H^1, L^2]_{1-s,2}$ [16].

For $z \in \mathbb{C}$ we define $e_z(x) = e^{izx} \in L^2(-\pi, \pi)$. Now suppose $\psi \in (H^1)^*$; we define its Fourier transform $F = \hat{\psi}$ to be the entire function $F(z) := \psi(e_z)$ for $z \in \mathbb{C}$. Let us first identify $(H^1)^*$ via its Fourier transform:

Proposition 3.1. *Let F be an entire function. In order that there exists $\psi \in (H^1)^*$ with $F = \hat{\psi}$ it is necessary and sufficient that:*

$$F \text{ is of exponential type } \leq \pi. \tag{3.1}$$

$$\int_{-\infty}^{\infty} \frac{|F(x)|^2}{1 + x^2} dx < \infty. \tag{3.2}$$

These conditions imply the estimate:

$$\sup_{z \in \mathbb{C}} \frac{|F(z)|}{(1 + |z|)e^{\pi|\Im z|}} < \infty. \tag{3.3}$$

Proof. These results follow immediately from the Paley-Wiener theorem once one observes that $\psi \in (H^1)^*$ if and only if ψ is of the form

$$\psi(f) = \alpha f(0) + \varphi(f')$$

where $\varphi \in (L^2)^*$. □

Consider H^1 with the inner product:

$$\langle f, g \rangle_t = (f', g') + t^2(f, g)$$

where $t > 0$. Let us denote by $\|\psi\|_t$ the norm of ψ with respect to $\|\cdot\|_t$ where $\|f\|_t^2 = \langle f, f \rangle_t$, i. e. $\|\psi\|_t := \sup\{|\psi(f)| : \|f\|_t \leq 1\}$. Set

$$s_0 = 1 - \lim_{\tau \to \infty} \sup_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}} \tag{3.4}$$

and

$$s_1 = 1 - \lim_{\tau \to \infty} \inf_{t \geq 1} \frac{1}{\log \tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}. \tag{3.5}$$

We can specialize Theorem (2.1) to the this special case of interpolating between L^2 and H^1.

Proposition 3.2. *Suppose $\psi \in (H^1)^*$ and let $Y_0 = \{f \in H^1 : \ \psi(f) = 0\}$. Then:*
(1) $(L^2, Y_0)_{s,2} = H^s$ if and only if $0 \le s < s_0$.
(2) $(L^2, Y_0)_{s,2}$ is a closed subspace of codimension one in H^s if and only if $s_1 < s \le 1$.

Proof. We can then apply Theorem 2.1 with $X_0 = H^1$ and $X_1 = L^2$. To estimate $K(t, \psi)$ we note that if $f \in H^1$ and $t \ge 1$ then

$$\max(\|f\|_1, t\|f\|) \le \|f\|_t \le \sqrt{2}\max(\|f\|_1, t\|f\|).$$

and so, for $t \ge 1$,

$$\|\psi\|_t \le K(t^{-1}, \psi) \le \sqrt{2}\|\psi\|_t.$$

Hence we can describe the numbers σ_0, σ_1 of Theorem 2.1 by

$$\sigma_1 = \lim_{\tau\to\infty} \sup_{t\ge 1} \frac{1}{\log\tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}$$

and

$$\sigma_0 = \lim_{\tau\to\infty} \inf_{t\ge 1} \frac{1}{\log\tau} \log \frac{\|\psi\|_t}{\|\psi\|_{\tau t}}.$$

Since $\sigma_1 = 1 - s_0$ and $\sigma_0 = 1 - s_1$ this proves the Proposition. □

We next turn to the problem of estimating $\|\psi\|_t$. The following lemma will be useful:

Lemma 3.3. *Suppose F satisfies (3.1) and (3.2). Then for any real t we have:*

$$|F(it)| \le |t|^{\frac{1}{2}} e^{\pi|t|} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|F(x)|^2}{t^2 + x^2} dx \right)^{\frac{1}{2}}. \tag{3.6}$$

Proof. It suffices to consider $t > 0$. Then by (3.3) $F(z)e^{i\pi z}(z+it)^{-1}$ is bounded and analytic in the upper-half plane and so we have:

$$F(it) = \frac{te^{\pi t}}{\pi} \int_{-\infty}^{\infty} \frac{F(x)}{x + it} \frac{e^{i\pi x}}{x - it}\, dx$$

Applying the Cauchy–Bunyakowski inequality we prove the lemma. □

We can now give an estimate for $\|\psi\|_t$ which essentially solve the problem of determining s_0 and s_1.

Theorem 3.4. *There exist a constant C so that for $t \ge 2$ we have*

$$\frac{1}{C} \left(\int_{-\infty}^{\infty} \frac{|F(x)|^2}{x^2 + t^2} dx \right)^{\frac{1}{2}} \le \|\psi\|_t \le C \left(\int_{-\infty}^{\infty} \frac{|F(x)|^2}{x^2 + t^2} dx \right)^{\frac{1}{2}} \tag{3.7}$$

.

Proof. We start with the remark that the functions $\{(2\pi)^{-\frac{1}{2}}(n^2+t^2)^{-\frac{1}{2}}e_n : n \in \mathbb{Z}\}$ together with $(\frac{1}{2}(t\sinh 2\pi t)^{-\frac{1}{2}}(e_{it}+e_{-it}))$ form an orthonormal basis of H^1 for $\|\cdot\|_t$. Hence

$$\|\psi\|_t^2 = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{|F(n)|^2}{n^2+t^2} + 4\frac{|F(it)+F(-it)|^2}{t\sinh 2\pi t}. \tag{3.8}$$

By (3.6) the last term in (3.8) can be estimated by

$$\frac{|F(it)+F(-it)|^2}{t\sinh 2\pi t} \le C^2\int_{-\infty}^{\infty}\frac{|F(x)|^2}{t^2+x^2}dx \tag{3.9}$$

for $t \ge 1$.

Now if $-1 \le \tau \le 1$ the map $T_\tau : H^1 \to H^1$ defined by $T_\tau f = e_\tau f$ satisfies $\|T_\tau\|_t \le 2$ provided $t \le 1$. Hence if $\psi_\tau = T_\tau^*\psi$ we have $\frac{1}{2}\|\psi_\tau\|_t \le \|\psi\|_t \le 2\|\psi_\tau\|_t$. However using (3.8) and (3.9) gives:

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{|F(n+\tau)|^2}{n^2+t^2} \le \|\psi_\tau\|_t^2 \le \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{|F(n+\tau)|^2}{n^2+t^2} + C^2\int_{-\infty}^{\infty}\frac{|F(x+\tau)|^2}{t^2+x^2}dx.$$

Now by integrating for $0 \le \tau \le 1$ we obtain (3.7). □

4. Application to nonharmonic Fourier series

At this point we turn our attention to exponential Riesz bases. Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers. For convenience we shall write $\sigma_n = \Re\lambda_n$ and $\tau_n = \Im\lambda_n$.

Let us suppose that $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2, or equivalently, $((1+|\tau_n|)^{\frac{1}{2}}e^{-\pi|\tau_n|}e_{\lambda_n})_{n\in\mathbb{Z}}$ is a Riesz basis of L^2. Then this family is *complete interpolating set* [25]. In particular, we have *sampling condition*: there exists a constant D so that if $f \in L^2$ then

$$D^{-1}\|f\| \le \left(\sum_{n\in\mathbb{Z}}(1+|\tau_n|)e^{-2\pi|\tau_n|}|\hat{f}(\lambda_n)|^2\right)^{\frac{1}{2}} \le D\|f\| \tag{4.1}$$

(i.e., the latter family is *a frame*). We also note that it must satisfy a *separation condition* i.e. for some $0 < \delta < 1$ we have:

$$\frac{|\lambda_m-\lambda_n|}{1+|\lambda_m-\overline{\lambda}_n|} \ge \delta \qquad m \ne n. \tag{4.2}$$

Then we can define an entire function F by

$$F(z) = \lim_{R\to\infty}\prod_{|\lambda_k|\le R}(1-\frac{z_k}{\lambda_k}). \tag{4.3}$$

We replace the term $(1-\lambda_k^{-1}z)$ by z if $\lambda_k = 0$. We call F the *generating function* for the unconditional basis (e_{λ_n}).

Proposition 4.1. [23, 14, 7] *The product (4.3) converges to an entire function of exponential type* π *and satisfies the integrability conditions (3.2) and*

$$\int_{-\infty}^{\infty} |F(x)|^2 dx = \infty. \tag{4.4}$$

Let us note, that the inequality in (3.2) is necessary for minimality of the family and (4.4) for completeness of (e_{λ_n}). Note that since F satisfies (3.1) and (3.2) so that there exists $\psi \in (H^1)^*$ with $\hat{\psi} = F$. We remark that F is a Cartwright class function and then [14] we have the Blaschke condition

$$\sum_{\lambda_n \neq 0} \frac{|\tau_n|}{|\lambda_n|^2} < \infty$$

Note that this implies that the families $(e_{\lambda_n})_{\Im \lambda_n > 0}$, $(e_{\lambda_n})_{\Im \lambda_n < 0}$ are minimal in $L^2(0, \infty)$, $L^2(-\infty, 0)$ correspondingly. Also we have $\sum_{\lambda_n \neq 0} \frac{1}{|\lambda_n|^2} < \infty$, (this follows from [15] p. 127). Thus, we have a strong Blaschke condition

$$\sum_{\lambda_n \neq 0} \frac{1 + |\tau_n|}{|\lambda_n|^2} < \infty. \tag{4.5}$$

Now by the result of Russell, Proposition 1.2, the functions (e_{λ_n}) form an unconditional basis of a closed subspace Y_0 of H^1 of codimension one. It is clear that the kernel of ψ coincides with Y_0. Hence our above results Proposition 3.2 and Theorem 3.4 apply to this case.

Theorem 4.2. *Suppose* $(e_{\lambda_n})_{n \in \mathbb{Z}}$ *is an unconditional basis of* L^2. *Then:*
(1) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ *is an unconditional basis of the Sobolev space* H^s *if and only if* $0 \leq s < s_0$.
(2) $(e_{\lambda_n})_{n \in \mathbb{Z}}$ *is an unconditional basis of a closed subspace of* H^s *of codimension one if and only if* $s_1 < s \leq 1$.
(3) If $s_0 \leq s \leq s_1$ *then* (e_{λ_n}) *is not an unconditional basic sequence.*

Proof. By Russell's theorem, Proposition 1.2 above, $(e_{\lambda_n})_{n \in \mathbb{Z}}$ is an unconditional basis for a closed subspace Y_0 of codimension one which is the kernel of the linear functional ψ. Let v_n be the weight sequence $v_n = \dfrac{\sinh(2\pi\tau_n)}{\tau_n} = \|e_{\lambda_n}\|_{L^2}^2$ and let $h_n = (1+|\lambda_n|^2)v_n = \|e_{\lambda_n}\|_{H^1}^2$. It follows from the basis property that the map $V : \ell_2(h) \to Y_0$ defined by

$$V(\alpha) = \sum_{n \in \mathbb{Z}} \alpha_n e_{\lambda_n}$$

is an isomorphism (onto). Clearly V is an isomorphism of $\ell_2(v)$ onto $L^2(-\pi, \pi) = Y_1$. Hence by interpolation V is an isomorphism of $\ell_2(v^{1-s}h^s)$ onto $Y_{1-s} =$

$[Y_0, L^2]_{1-s,2}$. In other words, setting $q_n = v_n^{1-s} h_n^s = v_n(1+|\lambda_n|^2)^s$, we have

$$C^{-1} \sum |\alpha_n|^2 q_n \leq \| \sum \alpha_n e_{\lambda_n} \|^2_{Y_{1-s}} \leq C \sum |\alpha_n|^2 q_n$$

and the almost normalized family $(e_{\lambda_n}/q_n^{1/2})_{n\in\mathbb{Z}}$ forms a Riesz basis in Y_{1-s}. Thus, if Y_{1-s} is a closed subspace in H^s, (e_{λ_n}) forms an unconditional basic sequence in H^s also.

We next estimate $\|e_{\lambda_n}\|_{H^s}$ to have the inverse implication. In fact from interpolation between L^2 and H^1 we have

$$\|e_{\lambda_n}\|_{H^s} \leq C \|e_{\lambda_n}\|^{1-s} \|e_{\lambda_n}\|_1^s = C(v_n^{1-s} h_n^s)^{\frac{1}{2}},$$

where C depends only on s. Similarly if we define $\phi_n(f) = (f, e_{\lambda_n})$ then the norm of ϕ_n in $(H^s)^*$ can be estimated by

$$\|\phi_n\|_{(H^s)^*} \leq C_1 \|\phi_n\|^{1-s} \|\phi_n\|^s_{(H^1)^*} = C_1 (v_n^{1-s})^{1/2} (v_n^2/h_n)^{s/2} = C_1 (v_n^{1+s} h_n^{-s})^{\frac{1}{2}}.$$

¿From the other hand, $\|e_{\lambda_n}\|_{H^s} \geq |\phi_n(e_{\lambda_n})|/\|\phi_n\|_{H^s}$, what gives $\|e_{\lambda_n}\|_{H^s} \geq C_1^{-1}(v_n^{1-s} h_n^s)^{1/2}$. Therefore the norms $\|e_{\lambda_n}\|_{H^s}$ and $\|e_{\lambda_n}\|_{Y_{1-s}}$ are both equivalent to $\sqrt{q_n}$. Therefore the assumption that (e_{λ_n}) is an unconditional basic sequence leads to equivalence of metrics H^s and Y_{1-s}. □

Remark. It is easy to have necessary and sufficient condition for an exponential family $(e_{\lambda_n})_{n\in\mathbb{Z}}$ which is complete and minimal in L^2 to be complete and/or minimal in H^s [3]. To do this we connect the generating function F with the critical exponent s_Λ

$$s_\Lambda := \inf\{s : \int_{-\infty}^{\infty} \frac{|F(x)|^2}{1+|x|^{2s}} dx < \infty = \inf\{s : \psi \in (H^s)^*\}.$$

Now $(e^{i\lambda_n t})$ is complete in $H^s(-\pi,\pi)$ for $s < s_\Lambda$ and is minimal for $s > s_\Lambda - 1$. The situation for $s = s_\Lambda$ or $s = s_\Lambda - 1$ depends on whether ψ is bounded in H^{s_Λ}. Note that $s_0 \leq s_\Lambda \leq s_1$ in general.

Thus, the family (e_{λ_n}) is minimal in H^s for $0 < s < 1$, and for any $(\alpha_n) \in l^2$, $\alpha \neq 0$,

$$0 < \| \sum \alpha_n e_{\lambda_n}/\sqrt{q_n} \|^2_{H^s} \leq \| \sum \alpha_n e_{\lambda_n}/\sqrt{q_n} \|^2_{Y_{1-s}} \leq C \sum |\alpha_n|^2.$$

We do not know whether $(e_{\lambda_n})_{n\in\mathbb{Z}}$ can be a conditional basis of H^s, for some appropriate ordering, when $s_0 \leq s \leq s_\Lambda$.

In order to get precise estimates of s_0 and s_1 we will need to establish an alternative formula for $\|\psi\|_t$ in this special case.

Let us introduce the function $\Phi(z)$ defined by $\Phi(z) = |F(z)| d(z,\Lambda)^{-1}$ when $z \notin \Lambda$ and $\Phi(\lambda_n) = |F'(\lambda_n)|$ for $n \in \mathbb{Z}$. The function Φ plays an important role in the known conditions for (e_{λ_n}) to be an unconditional basis (see [20] and [19]). We will call Φ the *carrier function* for (e_{λ_n}).

The following lemma lists some useful properties:

Lemma 4.3. *Suppose $-\infty < t < \infty$. Then:*
(i) There is at most one $n \in \mathbb{Z}$ so that $|it-\lambda_n| < \frac{1}{2}\delta|t|$ where δ is the separation constant in (4.2). There is also at most one $n \in \mathbb{Z}$ so that $|it-\lambda_n| < \frac{1}{4}\delta|\lambda_n|$.
(ii) $|F(it)| \le (|\lambda_0| + |t|)\Phi(it)$.
(iii) There is a constant C independent of t, n so that for every $n \in \mathbb{Z}$ we have:

$$|F(it)| \le C(|\lambda_0| + |t|)\frac{|it-\lambda_n|}{(|\lambda_n|^2+t^2)^{\frac{1}{2}}}\Phi(it).$$

(iv) We have for $t \neq 0$,

$$\Phi(it) \le |t|^{-\frac{1}{2}}e^{\pi|t|}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|F(x)|^2}{t^2+x^2}dx\right)^{\frac{1}{2}}.$$

Proof. (i) Suppose m, n are distinct and $|it-\lambda_n|, |it-\lambda_m| < \frac{1}{2}\delta|t|$. Then $|\lambda_m - \lambda_n| < \delta t$ while

$$|\lambda_m - \overline{\lambda_n}| \ge |(\lambda_m - it) - \overline{(\lambda_n - it)} + 2it| \ge (2-\delta)|t| > |t|.$$

Hence

$$\frac{|\lambda_m-\lambda_n|}{1+|\lambda_m-\overline{\lambda}_n|} < \delta$$

which contradicts (4.2).

For the second part note that if $|it-\lambda_n| < \frac{1}{4}\delta|\lambda_n|$ then $|\lambda_n| < 2|t|$ so that $|it-\lambda_n| < \frac{1}{2}\delta|t|$.

(ii) is immediate from the fact that $d(it,\Lambda) \le |\lambda_0| + t$.

(iii) If $|it-\lambda_n| < \frac{1}{2}\delta t$ then, in view of (i), $|it-\lambda_n|\Phi(it) = |F(it)|$ and $t^2+|\lambda_n|^2 \le 5t^2$. Let $|it-\lambda_n| \ge \frac{1}{2}\delta t$. Then

$$\frac{|it-\lambda_n|}{(|\lambda_n|^2+t^2)^{1/2}} \ge \frac{|it-\lambda_n|}{|\lambda_n|+|t|} \ge \frac{|it-\lambda_n|}{|it-\lambda_n|+2|t|} \ge c > 0.$$

Since $|\lambda_n| + |t| \ge d(it,\Lambda)$, we have (iii).

(iv) Let λ_n satisfy $|it-\lambda_n| = d(it,\Lambda)$. If t and τ_n have opposite signs or if $\tau_n = 0$, then $d(it,\Lambda) \ge t$ and so that $\Phi(it) \le t^{-1}|F(it)|$. If they have the same sign define $G(z) = (z-\overline{\lambda}_n)(z-\lambda_n)^{-1}F(z)$ and note that

$$\Phi(it) = \frac{|G(it)|}{|it-\overline{\lambda}_n|} \le t^{-1}|G(it)|.$$

Since $|G(x)| = |F(x)|$ for x real, we obtain (iv) from (3.6). $\square$

We next show that the Blaschke condition (4.5) can be improved for Riesz bases:

Proposition 4.4. *If $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2 then there is a constant C so that for any $0<t<\infty$,*

$$\sum_{\lambda_n\neq 0}\frac{t(1+|\tau_n|)}{|\lambda_n|^2+t^2}\le C. \tag{4.6}$$

Proof. Let us apply (4.1) to $e_{\pm it}$. Then

$$\sum_{n\in\mathbb{Z}}(1+|\tau_n|)e^{-2\pi|\tau_n|}\left|\frac{\sin(\pi(\lambda_n\pm it))}{\lambda_n\pm it}\right|^2\le 4D^2\|e_{\pm it}\|^2=4D^2\frac{\sinh 2\pi t}{t}. \tag{4.7}$$

Now for each n there is a choice of sign so that:

$$\left|\frac{\sin(\pi(\lambda_n\pm it))}{\lambda_n\pm it}\right|\ge\frac{|\sinh(\pi(|\tau_n|+t))|}{|\lambda_n|+t}$$

and hence

$$\sum_{n\in\mathbb{Z}}(1+|\tau_n|)e^{-2\pi|\tau_n|}\frac{\sinh(\pi(t+|\tau_n|)^2}{|\lambda_n|^2+t^2}\le 4D^2\frac{\sinh 2\pi t}{t}.$$

This yields (4.6) for $t\ge 1$ and this extends to $t\ge 0$ in view of (4.5) and the fact that $\sum_{n\neq 0}|\lambda_n|^{-2}<\infty$. $\square$

We will also need a perturbation lemma:

Lemma 4.5. *Let (e_{λ_n}) and $(e_{\mu_n})_{n\in\mathbb{Z}}$ be two unconditional bases of L^2. Suppose further that there is a constant C so that*

$$\sum_{n\in\mathbb{Z}}\frac{t|\mu_n-\lambda_n|}{|\mu_n||\lambda_n|+t^2}\le C\qquad 1<t<\infty.$$

Suppose Φ and Ψ are the carrier functions for (e_{λ_n}) and (e_{μ_n}). Then there exist constants $B,T>0$ so that if $t\ge T$

$$\frac{1}{B}\frac{\Psi(it)}{\Phi(it)}\le\prod_{\substack{0<|\lambda_n|\le t\\ |\mu_n|\neq 0}}\frac{|\lambda_n|}{|\mu_n|}\le B\frac{\Psi(it)}{\Phi(it)}.$$

Proof. We observe that for each n we have (taking $t=\max(1,|\mu_n|^{\frac12}|\lambda_n|^{\frac12})$)

$$|\lambda_n-\mu_n|\le C\max(1,2|\mu_n|^{\frac12}|\lambda_n|^{\frac12}).$$

Hence

$$|\lambda_n|\le|\mu_n|+2C|\lambda_n|^{1/2}|\mu_n|^{1/2}+C\le|\mu_n|+\frac12|\lambda_n|+2C^2|\mu_n|+C,$$

Along with a similar estimate for $|\mu_n|$ and setting $C_1=2+4C^2>1$ we get:

$$|\lambda_n|\le C_1(|\mu_n|+1),\quad |\mu_n|\le C_1(|\lambda_n|+1). \tag{4.8}$$

Now let $c = \frac{1}{4}\min(\delta, \delta')$ where δ, δ' are the separation constants of $(\lambda_n)_{n\in\mathbb{Z}}$ and $(\mu_n)_{n\in\mathbb{Z}}$ respectively.

We next make the remark that there is a constant M so that if $|w|, |z| \leq 2C_1 + 1$ and $|1-w|, |1-z| \geq c$ then

$$|\log|1-w| - \log|1-z|| \leq M|w-z|. \tag{4.9}$$

Let us fix $T = |\mu_0| + |\lambda_0| + 2C_1$. Suppose that $t \geq T$, and let $p = p(t), q = q(t) \in \mathbb{Z}$ be chosen so that $|it - \lambda_p| = \min\{|it - \lambda_n| : \ n \in \mathbb{Z}\}$ and $|it - \mu_q| = \min\{|it - \mu_n| : \ n \in \mathbb{Z}\}$. It may happen that $p = q$. Note that we have an automatic estimate,

$$|it - \lambda_p| \leq |t| + |\lambda_0| \leq 2t, \quad |it - \mu_q| \leq |t| + |\mu_0| \leq 2t. \tag{4.10}$$

Then if $n \neq p, q$ and $|\lambda_n| > t$ we have $|\mu_n| > \frac{1}{2}C_1^{-1}|\lambda_n|$ and so $|it - \mu_n| \leq |t| + |\mu_n| \leq (2C_1+1)|\mu_n|$. By Lemma 4.3 (i) we have $|it - \lambda_n| \geq c|\lambda_n|$ and $|it - \mu_n| \geq c|\mu_n|$. Hence we have by (4.9)

$$\begin{aligned} |\log|it-\mu_n| - \log|it-\lambda_n| - \log|\mu_n| + \log|\lambda_n|| &\leq M\frac{t|\lambda_n - \mu_n|}{|\lambda_n||\mu_n|} \\ &\leq (2C_1+1)M\frac{t|\lambda_n-\mu_n|}{|\lambda_n||\mu_n| + t^2}. \end{aligned}$$

Next suppose $n \neq p, q$ and $|\lambda_n| \leq t$. Then $|\mu_n| \leq C_1(t+1) \leq 2C_1 t$. We also have $|it - \lambda_n|, |it - \mu_n| \geq c|t|$ and so by (4.9)

$$\begin{aligned} |\log|it-\mu_n| - \log|it-\lambda_n|| &\leq M\frac{|\lambda_n - \mu_n|}{t} \\ &\leq (2C_1+1)M\frac{t|\lambda_n-\mu_n|}{|\lambda_n||\mu_n| + t^2}. \end{aligned}$$

Combining and summing over all $n \neq p, q$ we have

$$\log\frac{\Psi(it)}{\Phi(it)} = \delta(t)\log\left|\frac{it-\mu_p}{it-\lambda_q}\right| + \sum_{0<|\lambda_n|\leq t}\log|\lambda_n| - \sum_{\substack{0<|\lambda_n|\leq t \\ |\mu_n|\neq 0}}\log|\mu_n| + \gamma(t)$$

where $|\gamma(t)| \leq C(2C_1+1)M$ and $\delta(t) = 1$ if $p \neq q$ and 0 if $p = q$.

To conclude we need only consider the case $p \neq q$. In this case $|it - \mu_p|, |it - \lambda_q| \geq ct$. We also have $|\lambda_p|, |\mu_q| \leq 3t$ by (4.10) and so by (4.8) $|\lambda_q|, |\mu_p| \leq C_1(3t+1) \leq 4C_1 t$. Hence $|it - \mu_p|, |it - \lambda_q| \leq 5C_1 t$. This concludes the proof. $\square$

Lemma 4.6. *Suppose (e_{λ_n}) is an unconditional basis of L^2. Then there exist constants B, T so that if $t \geq T$ then*

$$\frac{1}{B}\Phi(it) \leq \Phi(-it) \leq B\Phi(it).$$

Proof. This follows from Lemma 4.5 taking $\mu_n = \overline{\lambda}_n$ in view of Lemma 4.4. $\square$

The next Theorem is the key step in the proof of our main result:

Theorem 4.7. *Suppose $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2. Then there is a constant C and $T>0$ so that if $t\geq T$ then*

$$C^{-1}t^{\frac{1}{2}}e^{-\pi t}\Phi(it)\leq \|\psi\|_t\leq Ct^{\frac{1}{2}}e^{-\pi t}\Phi(it). \tag{4.11}$$

Proof. The left-hand inequality in (4.11) is an immediate consequence of Lemma 4.3 (iv) and (3.7). We turn to the right-hand inequality.

We first use Lemma 4.3(ii), (iii) and Lemma 4.6. There are constants $C,T>1$ so that if $|t|\geq T$ we have $\Phi(-it)\leq C\Phi(it)$, $|F(it)|\leq Ct\Phi(it)$ and for every n,

$$|F(it)|\leq Ct\frac{|\lambda_n-it|}{(|\lambda_n|^2+t^2)^{\frac{1}{2}}}\Phi(it). \tag{4.12}$$

Choose $g\in H^1$ so that $\psi(f)=\langle f,g\rangle_t$ for $f\in H^1$. Let h be the orthogonal projection with respect to $\langle\cdot\rangle_t$ of g onto the subspace H_0^1 of all f so that $f(-\pi)=f(\pi)=0$ and let $k=g-h$. Then $\|\psi\|_t^2=\|k\|_t^2+\|h\|_t^2$.

The orthogonal complement of H_0^1 (with respect to $\langle\cdot\rangle_t$) is a 2-dimensional space with orthonormal basis $\{e_{\pm it}/\|e_{\pm it}\|_t\}$. Hence

$$k=\|e_{it}\|_t^{-2}(\overline{F(it)}e_{it}+\overline{F(-it)}e_{-it})$$

and

$$\|k\|_t^2=\|e_{it}\|_t^{-2}(|F(it)|^2+|F(-it)|^2).$$

Since $\|e_{it}\|_t^2=2t\sinh 2\pi t$, we deduce

$$\|k\|_t\leq C_1t^{\frac{1}{2}}\Phi(it)e^{-\pi t}\qquad t\geq T$$

and a suitable constant C_1. It remains therefore only to estimate $\|h\|_t$.

We first argue that

$$\begin{aligned}\langle e_z,k\rangle_t&=(2t\sinh 2\pi t)^{-1}(F(it)\langle e_z,e_{it}\rangle_t+F(-it)\langle e_z,e_{-it}\rangle_t)\\&=\frac{i}{\sinh 2\pi t}(F(-it)\sin\pi(z-it)-F(it)\sin\pi(z+it)).\end{aligned}$$

Since $\psi(e_{\lambda_n})=F(\lambda_n)=0$ for $n\in\mathbb{Z}$ we deduce that

$$\langle e_{\lambda_n},h\rangle_t=\frac{i}{\sinh 2\pi t}(F(it)\sin\pi(\lambda_n+it)-F(-it)\sin\pi(\lambda_n-it)).$$

Now if we use (4.12 we get an estimate valid for $t\geq T$:

$$|\langle e_{\lambda_n},h\rangle_t|\leq C\Phi(it)\frac{t|\lambda_n+it||\lambda_n-it|}{(|\lambda_n|^2+t^2)^{\frac{1}{2}}\sinh 2\pi t}\left(\left|\frac{\sin(\pi(\lambda_n-it))}{\lambda_n-it}\right|+\left|\frac{\sin(\pi(\lambda_n+it))}{\lambda_n+it}\right|\right).$$

Since $h\in H_0^1$ we then have

$$\langle e_{\lambda_n},h\rangle_t=(\lambda_n^2+t^2)(e_{\lambda_n},h)$$

and we can then rewrite the above estimate as

$$|(e_{\lambda_n}, h)| \le C\frac{\Phi(it)}{\sinh 2\pi t}\frac{1}{(|\lambda_n|^2+t^2)^{1/2}}\left(\left|\frac{\sin(\pi(\lambda_n - it))}{\lambda_n - it}\right| + \left|\frac{\sin(\pi(\lambda_n + it))}{\lambda_n + it}\right|\right).$$

Now

$$(e_{\lambda_n}, th + h') = (t - i\lambda_n)(e_{\lambda_n}, h).$$

We next use the sampling inequality (4.1):

$$\|h\|_t^2 = \|th + h'\|_{L^2}^2 \le D^2 \sum_{n\in\mathbb{Z}} (1+|\tau_n|)e^{-2\pi|\tau_n|}|t - i\lambda_n||(e_{\lambda_n}, h)|^2.$$

However we can combine with (4.7) to deduce that

$$\|th + h'\|_{L^2} \le 4C^2D^2t^{\frac12}\Phi(it)(\sinh 2\pi t)^{-\frac12}$$

for $t \ge T$ which gives the conclusion. $\square$

We now consider the case when (λ_n) is a small perturbation of the sequence $\mu_n = n$. For convenience we shall assume that $\lambda_n = 0$ can only occur when $n = 0$.

Theorem 4.8. *Suppose $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2 and for some constant C and all $t \ge 1$*

$$\sum_{n\neq 0} \frac{t|\lambda_n - n|}{n^2 + t^2} < C. \tag{4.13}$$

Then

$$s_1 = \frac12 + \lim_{\tau\to\infty}\sup_{t\ge 1}\frac{1}{\log\tau}\sum_{t<|n|\le\tau t}\log\frac{|n|}{|\lambda_n|}$$

and

$$s_0 = \frac12 + \lim_{\tau\to\infty}\inf_{t\ge 1}\frac{1}{\log\tau}\sum_{t<|n|\le\tau t}\log\frac{|n|}{|\lambda_n|}.$$

Proof. In this case we compare the carrier function Φ for the basis (e_{λ_n}) with the carrier function Ψ for the basis (e_n). Clearly $\Psi(it) = |\sin\pi it|/\pi t$. We can next use Lemma 4.5 to estimate $\Phi(it)$ and then the theorem follows directly from Theorem 4.7 together with (3.5) and (3.4). $\square$

Let us specialize to some important cases. Let $\delta_n = \Re\lambda_n - n = \sigma_n - n$.

Theorem 4.9. *Suppose $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2 such that $\sup|\delta_n| < \infty$ and $\sum_{n\neq 0}\tau_n^2 n^{-2} < \infty$. Then*

$$s_1 = \frac12 - \lim_{\tau\to\infty}\inf_{t\ge 1}\frac{1}{\log\tau}\sum_{t<|n|\le\tau t}\frac{\delta_n}{n} \tag{4.14}$$

and

$$s_0 = \frac{1}{2} - \lim_{\tau\to\infty} \sup_{t\geq 1} \frac{1}{\log \tau} \sum_{t<|n|\leq \tau t} \frac{\delta_n}{n}. \tag{4.15}$$

Remark. In particular (4.14) and (4.15) hold if $|\lambda_n - n|$ is bounded.

Proof. Combining Proposition 4.4 and the boundedness of (δ_n) gives us (4.13). Note that if $n \neq 0$,

$$\log \frac{|n|}{|\lambda_n|} = -\log(1 + \frac{|\lambda_n| - |n|}{|n|}).$$

Now

$$\begin{aligned}\frac{|\lambda_n| - |n|}{|n|} &= \left(1 + \frac{2\delta_n}{n} + \frac{\delta_n^2 + \tau_n^2}{n^2}\right)^{\frac{1}{2}} \\ &= \frac{\delta_n}{n} + \alpha_n\end{aligned}$$

where

$$|\alpha_n| \leq C\frac{1+\tau_n^2}{n^2}$$

for a suitable constant C. By (4.5) and the assumption of the theorem, this implies that $\sum_{n\neq 0} |\alpha_n| < \infty$ and yields the Theorem. $\square$

Before discussing examples we observe one more property of s_0 and s_1 in this case, which uses recent results of [19] and the theory of A_2-weights.

Theorem 4.10. *If $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basis of L^2 then $s_0 > 0$ and $s_1 < 1$.*

Proof. We will use the connections between Riesz basis property and sampling/interpolation in the spaces of entire functions of exponential type. These connections in the case of L^2 and the Paley–Wiener space may be found in [25].

Let $L^2_{\pi,s}$ consisting of all entire functions of exponential type at most π and satisfying

$$\int_{-\infty}^{\infty} \frac{|f(\xi)|^2}{(1+|\xi|)^{2s}} d\xi < \infty.$$

(Note that the Fourier transform of $L^2_{\pi,s}$ is the set of all distributions from $H^{-s}(\mathbb{R})$ supported on $[-\pi,\pi]$.) Now the formal adjoint of the map from $\ell_2(\mathbb{Z})$ to H^s defined by $(\alpha_n) \to \sum_{n\in\mathbb{Z}} \alpha_n(1+|\tau_n|)^{-\frac{1}{2}}(1+|\lambda_n|)^{-s} e_{\lambda_n}$ is the map from $L^2_{\pi,s}$ to $\ell_2(\mathbb{Z})$ given by $f \to \big(f(\lambda_n)(1+|\tau_n|)^{1/2}(1+|\lambda_n|)^s e^{-\pi\tau_n}\big)_{n\in\mathbb{Z}}$. Hence $(e_{\lambda_n})_{n\in\mathbb{Z}}$ is an unconditional basic sequence (resp. unconditional basis) if and only if $(\lambda_n)_{n\in\mathbb{Z}}$ is an interpolating sequence (resp. complete interpolating sequence) in $L^2_{\pi,s}$.

Note that if (λ_n) is interpolating for $L^2_{\pi,s-1}$ then it is interpolating for $L^2_{\pi,s}$ by the simple device of considering functions of the form $f(z)=(z-\mu)g(z)$ where $\mu\notin\Lambda=\{\lambda_n\}_{n\in\mathbb{Z}}$ and $g\in L^2_{\pi,s-1}$.

It follows that our result can be proved by showing that $(\lambda_n)_{n\in\mathbb{Z}}$ is a complete interpolating sequence for $L^2_{\pi,s}$ for all $|s|<\epsilon$ for some $\epsilon>0$. To do this we use the results of [19] that this is equivalent to requiring that $(1+|\xi|)^{2s}\Phi(\xi)^2$ is an A_2-weight for $|s|<\epsilon$. Now Φ^2 is an A_2-weight ([19] or [20]) and so there exists $\eta>0$ so that $\Phi^{2(1+\eta)}$ is an A_2-weight (cf. [6] p. 262, Corollary 6.10). Hence the Hilbert transform is bounded on both $L_2(\mathbb{R},\Phi^{2(1+\eta)})$ on $L^2(\mathbb{R},(1+|\xi|)^{2\theta})$ for $0<\theta<\frac{1}{2}$. It then follows by complex interpolation that $\Phi(\xi)^2(1+|\xi|)^{2s}$ is an A_2-weight when $|s|<\eta(1+\eta)^{-1}$. $\square$

Note that these results now imply Theorem 1.4.

Examples. We recall the classical theorem of Kadets, see, e.g., [11] or [15], that if (λ_n) are real then $\sup_n|\delta_n|<\frac{1}{4}$ is a sufficient condition for $(e_{\lambda_n})_{n\in\mathbb{Z}}$ to be a Riesz basis. First, we consider the case of regular behavior. For example, we can set $\delta_n=-\frac{1}{2}q\,\mathrm{sign}\,n$, see [1]. Then we obtain $s_1=s_0=\frac{1}{2}+q$. More generally if for some $y>0$ we have $C^{-1}(1+|x|)^{2q}\le|F(x+iy)|\le C(1+|x|)^{2q}$, we obtain $s_1=s_0=\frac{1}{2}+q$ (if we use the integral estimates of $\|\psi\|_t$, i.e. Theorem 3.4).

One can easily make sequences (δ_n) with $\sup|\delta_n|<\frac{1}{4}$ to exhibit any required behavior. In fact if we put

$$b_n=\frac{1}{\log 2}\sum_{2^n<|k|\le 2^{n+1}}\frac{\delta_k}{k}$$

then

$$s_0=\frac{1}{2}-\lim_{N\to\infty}\frac{1}{N}\inf_{n\ge1}\sum_{k=n+1}^{n+N}b_k$$

and

$$s_1=\frac{1}{2}-\lim_{N\to\infty}\frac{1}{N}\sup_{n\ge1}\sum_{k=n+1}^{n+N}b_k.$$

To be more specific if $-\frac{1}{2}<p\le q<\frac{1}{2}$, set

$$\begin{aligned}\delta_n&=\tfrac{1}{2}q\,\mathrm{sign}\,n &&\text{for } 2^{(2^{2k})}<|n|\le 2^{(2^{2k+1})}\\ \delta_n&=\tfrac{1}{2}p\,\mathrm{sign}\,n &&\text{for } 2^{(2^{2k-1})}<|n|\le 2^{(2^{2k})}\end{aligned}$$

Then for $2^{2k}<m\le 2^{2k+1}$

$$b_m=\frac{q}{\log 2}\sum_{2^m+1}^{2^{m+1}}\frac{1}{k}=q+o(1)$$

and for $2^{2k-1} < m \leq 2^{2k}$

$$b_m = p + o(1).$$

Thus

$$s_0 = \frac{1}{2} - q, \;\; s_1 = \frac{1}{2} - p$$

(note that an example of irregular behavior is given in [1])

References

[1] S.A. Avdonin, On Riesz bases of exponentials in L^2, *Vestnik Leningrad Univ., Ser. Mat., Mekh., Astron.*,(1974) (13):5–12, (Russian); English transl. in Vestnik Leningrad Univ. Math., v. 7 (1979), 203-211.

[2] S. Avdonin and S.A. Ivanov, *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge University Press, Cambridge, 1995.

[3] S. Avdonin and S. Ivanov. Levin–Golovin theorem for the Sobolev spaces. *Matemat. Zametki*, 68(2):163–172, 2000. (Russian); English transl. in Math. Notes.

[4] S. A. Avdonin, S. A. Ivanov, and D. L. Russell, Exponential bases in Sobolev spaces in control and observation problems for the wave equation, *Proceedings of the Royal Society of Edinburgh*, to appear.

[5] C. Bennett and R. Sharpley, *Interpolation of operators,* Academic Press, Orlando, 1988.

[6] J.B. Garnett, *Bounded analytic functions*, Academic Press, Orlando, 1981.

[7] S.A. Ivanov, Nonharmonic Fourier series in the Sobolev spaces of positive fractional orders, *New Zealand Journal of Mathematics*, 25 (1996) 36–46,

[8] S.A. Ivanov and B.S. Pavlov, Carleson series of resonances in the Regge problem, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 42(1) (1978) 26–55, (Russian); English transl. in Math. USSR Izvestija., v. 12, no. 1 (1978), 21–51.

[9] S. Janson, Interpolation of subcouples and quotient couples, *Ark. Mat.*, 31 (1993) 306–338.

[10] S. Kaijser and P. Sunehag, Interpolation of subspaces of codimension one, in preparation.

[11] S. V. Khrushchev, N. K. Nikol'skii, and B. S. Pavlov, *Unconditional bases of exponentials and reproducing kernels*, volume 864 of *Lecture Notes in Math."Complex Analysis and Spectral Theory"*. Springer–Verlag, Berlin/Heidelberg, 1981. pp. 214–335.

[12] N. Krugliak, L. Maligranda and L.E. Persson,The failure of the Hardy inequailty and interpolation of intersections,Ark. Mat. 37 (1999) 323–344.

[13] I. Lasiecka, J.-L. Lions, and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, *J. Math. Pures et Appl.*, 65(2) (1986) 149–192.

[14] B.Ya Levin, *Distribution of zeros of entire functions*, GITTL, Moscow, 1956, English transl., Amer.Math.Soc., Providence, RI,1964.

[15] B. Ya. Levin, *Lectures on entire functions,* Translations of Mathematical Monographs, Vol. 150, Amer. Math. Soc., Providence 1996.

[16] J.-L. Lions and E. Magenes, *Problèmes aux limites nonhomogénes et applications*, volume I,II. Dunod, Paris, 1968.

[17] J. Löfström, Real interpolation with constraints, J. Approx. Theory 82 (1995), 30–53.

[18] J. Löfström, Interpolation of subspaces, Preprint, University of Goteborg, 1997 http://www.math.chalmers.se/˜jorgen.

[19] Yu. Lyubarskii and K. Seip, Weighted Paley-Wiener spaces, submitted

[20] A.M. Minkin, Reflection of exponents, and unconditional bases of exponentials, *Algebra i Anal.*, 3(5) (1992) 110–135. (Russian); English transl. in St. Petersburg Math. J., v. 3, no. 5 (1992), 1043–1068.
[21] K. Narukawa and T. Suzuki, Nonharmonic Fourier series and its applications, *Appl. Math. Optim.*, 14 (1986) 249–264.
[22] N. K. Nikol'skiĭ, *A Treatise on the Shift Operator*, Springer, Berlin, 1986.
[23] R.E.A.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, volume XIX, AMS Coll. Publ., New York, 1934.
[24] D.L. Russell, On exponential bases for the Sobolev spaces over an interval, *J. Math. Anal. Appl.*, 87(2) (1982) 528–550.
[25] K. Seip, On the connection between exponential bases and certain related sequences in $L^2(-\pi,\pi)$, *J. of Functional Analysis*, 130(1) (1995) 131–160.
[26] R. Wallsten, Remarks on interpolation of subspaces, *Lecture Notes in Math.*, 1302 (1988) 410–419.

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